Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section – A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.

Special Instructions/Useful Data	
$\mathbb{N}$	Set of all natural numbers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers
P^T	Transpose of the matrix P
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n)^T \mid x_i \in \mathbb{R}, i = 1, 2, \dots, n\}$
g'	Derivative of a real valued function g
g''	Second derivative of a real valued function g
$P(A)$	Probability of an event A
i.i.d.	Independently and identically distributed
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$F_{m,n}$	F distribution with (m, n) degrees of freedom
t_n	Student's t distribution with n degrees of freedom
χ_n^2	Central Chi-squared distribution with n degrees of freedom
$\Phi(x)$	Cumulative distribution function of $N(0,1)$
A^C	Complement of a set A
$E(X)$	Expectation of a random variable X
$\text{Var}(X)$	Variance of a random variable X
$\text{Cov}(X, Y)$	Covariance between random variables X and Y
$r!$	Factorial of an integer $r > 0$, $0! = 1$
$\Phi(0.25) = 0.5987, \Phi(0.5) = 0.6915, \Phi(0.625) = 0.7341, \Phi(0.71) = 0.7612,$ $\Phi(1) = 0.8413, \Phi(1.125) = 0.8697, \Phi(1.5) = 0.9332, \Phi(1.64) = 0.95,$ $\Phi(2) = 0.9772$	

SECTION – A

MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 The imaginary parts of the eigenvalues of the matrix

$$P = \begin{pmatrix} 3 & 2 & 5 \\ 2 & -3 & 6 \\ 0 & 0 & -3 \end{pmatrix}$$

are

(A) $0, 0, 0$ (B) $2, -2, 0$ (C) $1, -1, 0$ (D) $3, -3, 0$

Q.2 Let $u, v \in \mathbb{R}^4$ be such that $u = (1 \quad 2 \quad 3 \quad 5)^T$ and $v = (5 \quad 3 \quad 2 \quad 1)^T$. Then the equation $uv^T x = v$ has

(A) infinitely many solutions (B) no solution

(C) exactly one solution (D) exactly two solutions

Q.3 Let $u_n = \left(4 - \frac{1}{n}\right)^{\frac{(-1)^n}{n}}$, $n \in \mathbb{N}$ and let $l = \lim_{n\to\infty} u_n$.

Which of the following statements is TRUE?

(A) $l = 0$ and $\sum_{n=1}^{\infty} u_n$ is convergent

(B) $l = \frac{1}{4}$ and $\sum_{n=1}^{\infty} u_n$ is divergent

(C) $l = \frac{1}{4}$ and $\{u_n\}_{n\geq 1}$ is oscillatory

(D) $l = 1$ and $\sum_{n=1}^{\infty} u_n$ is divergent

Q.4 Let $\{a_n\}_{n\geq 1}$ be a sequence defined as follows:

$$a_1 = 1 \text{ and } a_{n+1} = \frac{7a_n + 11}{21}, n \in \mathbb{N}.$$

Which of the following statements is TRUE?

(A) $\{a_n\}_{n\geq 1}$ is an increasing sequence which diverges

(B) $\{a_n\}_{n\geq 1}$ is an increasing sequence with $\lim_{n\to\infty} a_n = \frac{11}{14}$

(C) $\{a_n\}_{n\geq 1}$ is a decreasing sequence which diverges

(D) $\{a_n\}_{n\geq 1}$ is a decreasing sequence with $\lim_{n\to\infty} a_n = \frac{11}{14}$

Q.5 Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x^3, & \text{if } 0 < x \leq 1 \\ \frac{3}{x^5}, & \text{if } x > 1 \end{cases}.$$

Then $P\left(\frac{1}{2} < X < 2\right)$ equals

(A) $\frac{15}{16}$ (B) $\frac{11}{16}$ (C) $\frac{7}{12}$ (D) $\frac{3}{8}$

Q.6 Let X be a random variable with the moment generating function

$$M_X(t) = \frac{1}{216}(5 + e^t)^3, \quad t \in \mathbb{R}.$$

Then $P(X > 1)$ equals

(A) $\frac{2}{27}$ (B) $\frac{1}{27}$ (C) $\frac{1}{12}$ (D) $\frac{2}{9}$

Q.7 Let X be a discrete random variable with the probability mass function

$$p(x) = k(1 + |x|)^2, \quad x = -2, -1, 0, 1, 2,$$

where k is a real constant. Then $P(X = 0)$ equals

(A) $\frac{1}{9}$ (B) $\frac{2}{27}$ (C) $\frac{1}{27}$ (D) $\frac{1}{81}$

Q.8 Let the random variable X have uniform distribution on the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. Then $P(\cos X > \sin X)$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Q.9 Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables having common probability density function

$$f(x) = \begin{cases} xe^{-x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n = 1,2, \ldots$. Then $\lim_{n \to \infty} P(\bar{X}_n = 2)$ equals

(A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1

Q.10 Let X_1, X_2, X_3 be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.$$

Which of the following estimators of θ has the smallest variance for all $\theta > 0$?

(A) $\frac{X_1+3X_2+X_3}{5}$ (B) $\frac{X_1+X_2+2X_3}{4}$

(C) $\frac{X_1+X_2+X_3}{3}$ (D) $\frac{X_1+2X_2+3X_3}{6}$

Q. 11 – Q. 30 carry two marks each.

Q.11 Player P_1 tosses 4 fair coins and player P_2 tosses a fair die independently of P_1. The probability that the number of heads observed is more than the number on the upper face of the die, equals

(A) $\frac{7}{16}$ (B) $\frac{5}{32}$ (C) $\frac{17}{96}$ (D) $\frac{21}{64}$

Q.12 Let X_1 and X_2 be i.i.d. continuous random variables with the probability density function

$$f(x) = \begin{cases} 6\,x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Using Chebyshev's inequality, the lower bound of $P\left(|X_1 + X_2 - 1| \le \frac{1}{2}\right)$ is

(A) $\frac{5}{6}$ (B) $\frac{4}{5}$ (C) $\frac{3}{5}$ (D) $\frac{1}{3}$

Q.13 Let X_1, X_2, X_3 be i.i.d. discrete random variables with the probability mass function

$$p(k) = \left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right), \quad k = 1, 2, 3, \ldots$$

Let $Y = X_1 + X_2 + X_3$. Then $P(Y \ge 5)$ equals

(A) $\frac{1}{9}$ (B) $\frac{8}{9}$ (C) $\frac{2}{27}$ (D) $\frac{25}{27}$

Q.14 Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} cx(1-x), & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases},$$

where c is a positive real constant. Then $E(X)$ equals

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{2}{5}$ (D) $\frac{1}{3}$

Q.15 Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} x+y, & \text{if } 0<x<1, 0<y<1 \\ 0, & \text{otherwise} \end{cases}.$$

Then $P\left(X+Y>\frac{1}{2}\right)$ equals

(A) $\frac{23}{24}$ (B) $\frac{1}{12}$ (C) $\frac{11}{12}$ (D) $\frac{1}{24}$

Q.16 Let $X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_n$ be i.i.d. $N(0,1)$ random variables. Then

$$W = \frac{n\left(\sum_{i=1}^{m} X_i\right)^2}{m\left(\sum_{j=1}^{n} Y_j^2\right)}$$

has

(A) χ^2_{m+n} distribution (B) t_n distribution

(C) $F_{m,n}$ distribution (D) $F_{1,n}$ distribution

Q.17 Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with the probability mass function

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 4 \\ \frac{3}{4}, & \text{if } x = 8 \\ 0, & \text{otherwise} \end{cases}.$$

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n = 1,2,\dots$. If $\lim_{n\to\infty} P(m \leq \bar{X}_n \leq M) = 1$, then possible values of m and M are

(A) $m = 2.1,\ M = 3.1$ (B) $m = 3.2,\ M = 4.1$

(C) $m = 4.2,\ M = 5.7$ (D) $m = 6.1,\ M = 7.1$

Q.18 Let $x_1 = 1.1,\ x_2 = 0.5,\ x_3 = 1.4,\ x_4 = 1.2$ be the observed values of a random sample of size four from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} e^{\theta - x}, & \text{if } x \geq \theta \\ 0, & \text{otherwise} \end{cases}, \quad \theta \in (-\infty, \infty).$$

Then the maximum likelihood estimate of θ^2 is

(A) 0.5 (B) 0.25 (C) 1.21 (D) 1.44

Q.19 Let $x_1 = 2,\ x_2 = 1,\ x_3 = \sqrt{5},\ x_4 = \sqrt{2}$ be the observed values of a random sample of size four from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{2\theta}, & \text{if } -\theta \le x \le \theta \\ 0, & \text{otherwise} \end{cases}, \qquad \theta > 0.$$

Then the method of moments estimate of θ is

(A) 1 (B) 2 (C) 3 (D) 4

Q.20 Let $X_1,\ X_2$ be a random sample from an $N(0,\theta)$ distribution, where $\theta > 0$. Then the value of k, for which the interval $\left(0, \frac{X_1^2+X_2^2}{k}\right)$ is a 95% confidence interval for θ, equals

(A) $-\log_e(0.95)$ (B) $-2\log_e(0.95)$ (C) $-\frac{1}{2}\log_e(0.95)$ (D) 2

Q.21 Let $X_1,\ X_2,\ X_3,\ X_4$ be a random sample from $N(\theta_1,\sigma^2)$ distribution and $Y_1,\ Y_2,\ Y_3,\ Y_4$ be a random sample from $N(\theta_2,\sigma^2)$ distribution, where $\theta_1,\theta_2 \in (-\infty,\infty)$ and $\sigma > 0$. Further suppose that the two random samples are independent. For testing the null hypothesis $H_0: \theta_1 = \theta_2$ against the alternative hypothesis $H_1: \theta_1 > \theta_2$, suppose that a test ψ rejects H_0 if and only if $\sum_{i=1}^{4} X_i > \sum_{j=1}^{4} Y_j$. The power of the test ψ at $\theta_1 = 1+\sqrt{2},\ \theta_2 = 1$ and $\sigma^2 = 4$ is

(A) 0.5987 (B) 0.7341 (C) 0.7612 (D) 0.8413

Q.22 Let X be a random variable having a probability density function $f \in \{f_0, f_1\}$, where

$$f_0(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_1(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

For testing the null hypothesis $H_0: f \equiv f_0$ against $H_1: f \equiv f_1$, based on a single observation on X, the power of the most powerful test of size $\alpha = 0.05$ equals

(A) 0.425 (B) 0.525 (C) 0.625 (D) 0.725

Q.23 If

$$\int_{y=0}^{1} \int_{x=y}^{2-\sqrt{1-(y-1)^2}} f(x,y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{\alpha(x)} f(x,y)\,dy\,dx + \int_{x=1}^{2} \int_{y=0}^{\beta(x)} f(x,y)\,dy\,dx,$$

then $\alpha(x)$ and $\beta(x)$ are

(A) $\alpha(x) = x,\ \beta(x) = 1 + \sqrt{1-(x-2)^2}$ (B) $\alpha(x) = x,\ \beta(x) = 1 - \sqrt{1-(x-2)^2}$

(C) $\alpha(x) = 1 + \sqrt{1-(x-2)^2},\ \beta(x) = x$ (D) $\alpha(x) = 1 - \sqrt{1-(x-2)^2},\ \beta(x) = x$

Q.24 Let $f: [0,1] \to \mathbb{R}$ be a function defined as

$$f(t) = \begin{cases} t^3\left(1 + \frac{1}{5}\cos(\log_e t^4)\right) & \text{if } t \in (0,1] \\ 0 & \text{if } t = 0 \end{cases}.$$

Let $F: [0,1] \to \mathbb{R}$ be defined as

$$F(x) = \int_0^x f(t)dt.$$

Then $F''(0)$ equals

(A) 0 (B) $\frac{3}{5}$ (C) $-\frac{5}{3}$ (D) $\frac{1}{5}$

Q.25 Consider the function

$$f(x,y) = x^3 - y^3 - 3x^2 + 3y^2 + 7, \; x,y \in \mathbb{R}.$$

Then the local minimum (m) and the local maximum (M) of f are given by

(A) $m = 3,\ M = 7$ (B) $m = 4,\ M = 11$

(C) $m = 7,\ M = 11$ (D) $m = 3,\ M = 11$

Q.26 For $c \in \mathbb{R}$, let the sequence $\{u_n\}_{n\geq 1}$ be defined by

$$u_n = \frac{\left(1 + \frac{c}{n}\right)^{n^2}}{\left(3 - \frac{1}{n}\right)^n}.$$

Then the values of c for which the series $\sum_{n=1}^{\infty} u_n$ converges are

(A) $\log_e 6 < c < \log_e 9$ (B) $c < \log_e 3$

(C) $\log_e 9 < c < \log_e 12$ (D) $\log_e 3 < c < \log_e 6$

Q.27 If for a suitable $\alpha > 0$,

$$\lim_{x\to 0}\left(\frac{1}{e^{2x} - 1} - \frac{1}{\alpha x}\right)$$

exists and is equal to l $(|\,l\,| < \infty)$, then

(A) $\alpha = 2,\ l = 2$ (B) $\alpha = 2,\ l = -\frac{1}{2}$

(C) $\alpha = \frac{1}{2}, l = -2$ (D) $\alpha = \frac{1}{2},\ l = \frac{1}{2}$

Q.28 Let

$$P = \int_0^1 \frac{dx}{\sqrt{8 - x^2 - x^3}}.$$

Which of the following statements is TRUE?

(A) $\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) < P < \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{2}\right)$

(B) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{2}\right) < P < \sin^{-1}\left(\frac{1}{2}\right)$

(C) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) < P < \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

(D) $\sin^{-1}\left(\frac{1}{2}\right) < P < \frac{\sqrt{3}}{2}\sin^{-1}\left(\frac{1}{2}\right)$

Q.29 Let Q, A, B be matrices of order $n \times n$ with real entries such that Q is orthogonal and A is invertible. Then the eigenvalues of $Q^T A^{-1} B Q$ are always the same as those of

(A) AB

(B) $Q^T A^{-1} B$

(C) $A^{-1} B Q^T$

(D) BA^{-1}

Q.30 Let $(x(t), y(t))$, $1 \le t \le \pi$, be the curve defined by

$$x(t) = \int_1^t \frac{\cos z}{z^2}\, dz \quad \text{and} \quad y(t) = \int_1^t \frac{\sin z}{z^2}\, dz \ .$$

Let L be the length of the arc of this curve from the origin to the point P on the curve at which the tangent is perpendicular to the x-axis. Then L equals

(A) $\sqrt{2}$

(B) $\frac{\pi}{\sqrt{2}}$

(C) $1 - \frac{2}{\pi}$

(D) $\frac{\pi}{2} + \sqrt{2}$

SECTION - B
MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 Let $v \in \mathbb{R}^k$ with $v^T v \neq 0$. Let

$$P = I - 2\frac{vv^T}{v^T v},$$

where I is the $k \times k$ identity matrix. Then which of the following statements is (are) TRUE?

(A) $P^{-1} = I - P$

(B) -1 and 1 are eigenvalues of P

(C) $P^{-1} = P$

(D) $(I + P)v = v$

Q.32 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences of real numbers such that $\{a_n\}_{n\geq 1}$ is increasing and $\{b_n\}_{n\geq 1}$ is decreasing. Under which of the following conditions, the sequence $\{a_n + b_n\}_{n\geq 1}$ is always convergent?

(A) $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded sequences

(B) $\{a_n\}_{n\geq 1}$ is bounded above

(C) $\{a_n\}_{n\geq 1}$ is bounded above and $\{b_n\}_{n\geq 1}$ is bounded below

(D) $a_n \to \infty$ and $b_n \to -\infty$

Q.33 Let $f: [0,1] \to [0,1]$ be defined as follows:

$$f(x) = \begin{cases} x\,, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ x + \frac{2}{3}, & \text{if } x \in \mathbb{Q}^C \cap \left(0, \frac{1}{3}\right) \\ x - \frac{1}{3}\,, & \text{if } x \in \mathbb{Q}^C \cap \left(\frac{1}{3}, 1\right) \end{cases}.$$

Which of the following statements is (are) TRUE?

(A) f is one-one and onto

(B) f is not one-one but onto

(C) f is continuous on $\mathbb{Q} \cap [0,1]$

(D) f is discontinuous everywhere on $[0,1]$

Q.34 Let $f(x)$ be a nonnegative differentiable function on $[a,b] \subset \mathbb{R}$ such that $f(a) = 0 = f(b)$ and $|f'(x)| \leq 4$. Let L_1 and L_2 be the straight lines given by the equations $y = 4(x - a)$ and $y = -4(x - b)$, respectively. Then which of the following statements is (are) TRUE?

(A) The curve $y = f(x)$ will always lie below the lines L_1 and L_2

(B) The curve $y = f(x)$ will always lie above the lines L_1 and L_2

(C) $\left| \int_a^b f(x)dx \right| < (b-a)^2$

(D) The point of intersection of the lines L_1 and L_2 lie on the curve $y = f(x)$

Q.35 Let E and F be two events with $0 < P(E) < 1$, $0 < P(F) < 1$ and $P(E) + P(F) \geq 1$. Which of the following statements is (are) TRUE?

(A) $P(E^C) \leq P(F)$

(B) $P(E \cup F) < P(E^C \cup F^C)$

(C) $P(E|F^C) \geq P(F^C|E)$

(D) $P(E^C|F) \leq P(F|E^C)$

Q.36 The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{4}{9}, & \text{if } 0 \le x < 1 \\ \frac{8}{9}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$

Which of the following statements is (are) TRUE?

(A) The random variable X takes positive probability only at two points

(B) $P(1 \le X \le 2) = \frac{5}{9}$

(C) $E(X) = \frac{2}{3}$

(D) $P(0 < X < 1) = \frac{4}{9}$

Q.37 Let X_1, X_2 be a random sample from a distribution with the probability mass function

$$f(x|\theta) = \begin{cases} 1-\theta, & \text{if } x = 0 \\ \theta, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}, \qquad 0 < \theta < 1.$$

Which of the following is (are) unbiased estimator(s) of θ?

(A) $\frac{X_1+X_2}{2}$ (B) $\frac{X_1^2+X_2}{2}$ (C) $\frac{X_1^2+X_2^2}{2}$ (D) $\frac{X_1+X_2-X_1^2}{2}$

Q.38 Let X_1, X_2, X_3 be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \qquad \theta > 0.$$

If $\delta(X_1, X_2, X_3)$ is an unbiased estimator of θ, which of the following CANNOT be attained as a value of the variance of δ at $\theta = 1$?

(A) 0.1 (B) 0.2 (C) 0.3 (D) 0.5

Q.39 Let $X_1, X_2, \ldots, X_n$ $(n \ge 2)$ be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{x}{\theta^2} e^{-x/\theta}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \qquad \theta > 0.$$

Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Which of the following statistics is (are) sufficient but NOT complete?

(A) $\bar{X}$ (B) $\bar{X}^2 + 3$ (C) $(X_1, \sum_{i=2}^{n} X_i)$ (D) $(X_1, \bar{X})$

Q.40 Let X_1, X_2, X_3, X_4 be a random sample from an $N(\theta, 1)$ distribution, where $\theta \in (-\infty, \infty)$. Suppose the null hypothesis $H_0: \theta = 1$ is to be tested against the hypothesis $H_1: \theta < 1$ at $\alpha = 0.05$ level of significance. For what observed values of $\sum_{i=1}^{4} X_i$, the uniformly most powerful test would reject H_0?

(A) -1 (B) 0 (C) 0.5 (D) 0.8

SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 Let the random variable X have uniform distribution on the interval (0, 1) and $Y = -2\log_e X$. Then $E(Y)$ equals ______

Q.42 If $Y = \log_{10} X$ has $N(\mu, \sigma^2)$ distribution with moment generating function $M_Y(t) = e^{5t+2t^2}$, $t \in (-\infty, \infty)$, then $P(X < 1000)$ equals ________________

Q.43 Let X_1, X_2, X_3, X_4, X_5 be independent random variables with $X_1 \sim N(200, 8)$, $X_2 \sim N(104, 8)$, $X_3 \sim N(108, 15)$, $X_4 \sim N(120, 15)$ and $X_5 \sim N(210, 15)$. Let $U = \frac{X_1+X_2}{2}$ and $V = \frac{X_3+X_4+X_5}{3}$. Then $P(U > V)$ equals ________________

Q.44 Let X and Y be discrete random variables with the joint probability mass function

$$p(x,y) = \frac{1}{25}(x^2 + y^2), \quad \text{if } x = 1,2\,; y = 0,1,2.$$

Then $P(Y = 1 \mid X = 1)$ equals ________________

Q.45 Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} 8xy, & 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then $9\mathrm{Cov}(X, Y)$ equals ________________

Q.46 Let $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$ be i.i.d. $N(\mu, \sigma^2)$ random variables. Let $\bar{X} = \frac{1}{3}\sum_{i=1}^{3} X_i$ and $\bar{Y} = \frac{1}{4}\sum_{j=1}^{4} Y_j$. If $k\sqrt{\frac{15}{7}} \frac{(\bar{X} - \bar{Y})}{\sqrt{\left\{\sum_{i=1}^{3}(X_i - \bar{X})^2 + \sum_{j=1}^{4}(Y_j - \bar{Y})^2\right\}}}$ has t_ν distribution, then $(\nu - k)$ equals

____

Q.47 Let $f: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ be defined as

$$f(x) = \alpha x + \beta \sin x,$$

where $\alpha, \beta \in \mathbb{R}$. Let f have a local minimum at $x = \frac{\pi}{4}$ with $f\left(\frac{\pi}{4}\right) = \frac{\pi - 4}{4\sqrt{2}}$.

Then $8\sqrt{2}\,\alpha + 4\,\beta$ equals ______________

Q.48 The area bounded between two parabolas $y = x^2 + 4$ and $y = -x^2 + 6$ is ______________

Q.49 For $j = 1,2,\ldots,5$, let P_j be the matrix of order 5×5 obtained by replacing the j^{th} column of the identity matrix of order 5×5 with the column vector $v = (5 \quad 4 \quad 3 \quad 2 \quad 1)^T$. Then the determinant of the matrix product $P_1 P_2 P_3 P_4 P_5$ is ______________

Q.50 Let

$$u_n = \frac{18n + 3}{(3n-1)^2(3n+2)^2}, \qquad n \in \mathbb{N}.$$

Then $\sum_{n=1}^{\infty} u_n$ equals ______________

Q. 51 – Q. 60 carry two marks each.

Q.51 Let a unit vector $v = (v_1 \quad v_2 \quad v_3)^T$ be such that $Av = 0$ where

$$A = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}.$$

Then the value of $\sqrt{6}\,(|v_1| + |v_2| + |v_3|)$ equals ______________

Q.52 Let

$$F(x) = \int_0^x e^t (t^2 - 3t - 5)\,dt, \qquad x > 0.$$

Then the number of roots of $F(x) = 0$ in the interval $(0, 4)$ is ______________

Q.53 A tangent is drawn on the curve $y = \frac{1}{3}\sqrt{x^3}$, $(x > 0)$ at the point $P\left(1, \frac{1}{3}\right)$ which meets the x-axis at Q. Then the length of the closed curve $OQPO$, where O is the origin, is __________________

Q.54 The volume of the region

$$R = \{(x, y, z) \in \mathbb{R}^3 : x + y + z \leq 3,\ y^2 \leq 4x, 0 \leq x \leq 1, y \geq 0, z \geq 0\}$$

is ______________________________

Q.55 Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{x}{8}, & \text{if } 0 < x < 2 \\ \frac{k}{8}, & \text{if } 2 \leq x \leq 4 \\ \frac{6-x}{8}, & \text{if } 4 < x < 6 \\ 0, & \text{otherwise.} \end{cases},$$

where k is a real constant. Then $P(1 < X < 5)$ equals __________________

Q.56 Let X_1, X_2, X_3 be independent random variables with the common probability density function

$$f(x) = \begin{cases} 2e^{-2x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Let $Y = \min\{X_1, X_2, X_3\}, E(Y) = \mu_y$ and $\mathrm{Var}(Y) = \sigma_y^2$. Then $P(Y > \mu_y + \sigma_y)$ equals __________________

Q.57 Let X and Y be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{2}e^{-x}, & \text{if } |y| \leq x,\ x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Then $E(X \mid Y = -1)$ equals __________________

Q.58 Let X and Y be discrete random variables with $P(Y \in \{0,1\}) = 1$,

$$P(X = 0) = \frac{3}{4}, \qquad P(X = 1) = \frac{1}{4},$$
$$P(Y = 1|X = 1) = \frac{3}{4}, \quad P(Y = 0|X = 0) = \frac{7}{8}.$$

Then $3P(Y = 1) - P(Y = 0)$ equals __________________

Q.59 Let $X_1, X_2, \ldots, X_{100}$ be i.i.d. random variables with $E(X_1) = 0$, $E(X_1^2) = \sigma^2$, where $\sigma > 0$. Let $S = \sum_{i=1}^{100} X_i$. If an approximate value of $P(S \leq 30)$ is 0.9332, then σ^2 equals______________

Q.60 Let X be a random variable with the probability density function

$$f(x|r,\lambda) = \frac{\lambda^r}{(r-1)!} x^{r-1} e^{-\lambda x}, \quad x > 0, \lambda > 0, r > 0.$$

If $E(X) = 2$ and $\text{Var}(X) = 2$, then $P(X < 1)$ equals ______________

END OF THE QUESTION PAPER